Lecture 3

Monday, October 7, 2019 6:17 AM

__Recall__: • A metric space (X, d) is connected if $X, \emptyset$ are the only subsets that are both open and closed.

• Equivalently, X is not connected if $X = A \cup B$, $A \cap B = \emptyset$ and both A, B open.

__Note__: If A, B open, $A \cap B = \emptyset$, then $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. (Why?)

__Correction__. In Ex. from end of last lecture: $A, B \subseteq \mathbb{C}$, $A \cap B = \emptyset \Rightarrow$ $A \cup B$ not connected needs additional assumptions. Either: (1) A, B open in $\mathbb{C}$, or (2) $\overline{A} \cap B = A \cap \overline{B} = \emptyset$.

[Finish material from Lecture 2 notes.]

__Def.__ A __component__ of X is a maximal connected subset; i.e., $D \subseteq X$ is a component if D is connected and there are no connected subsets that contain D as a proper subset.

__Thm 1__. If (X, d) is a metric space, then $X = \bigcup_{\alpha \in I} D_\alpha$ where $\{D_\alpha\}_{\alpha \in I}$ is the collection of all components. Moreover, $D_\alpha \cap D_\beta = \emptyset$ if $\alpha \neq \beta$.

__Thm 2__. Let $G \subseteq \mathbb{C}$ be open. Then the collection of components is countable, and each component is open.

The proofs hinge on the following simple lemma:

__Lemma 1__. Let $x_0 \in X$ and $\{D_\alpha\}_{\alpha \in I}$ a collection of connected subsets such that $x_0 \in \bigcap_{\alpha \in I} D_\alpha$. Then $\bigcup_{\alpha \in I} D_\alpha$ is connected.

__Pf:__ Suppose $A \subseteq \bigcup_{\alpha \in I} D_\alpha$ is open, closed, $\neq \emptyset$. Must show $A = \bigcup D_\alpha$. For each α, $A \cap D_\alpha$ is open and closed in D_α. Since D_α is connected, either $A \cap D_\alpha = \emptyset$ or $A \cap D_\alpha = D_\alpha$. But $A \neq \emptyset$, so $A \cap D_\beta \neq \emptyset$ for some $\beta \Rightarrow A \cap D_\beta = D_\beta \Rightarrow x_0 \in A \Rightarrow A \cap D_\alpha \neq \emptyset$ $\forall \alpha$

or $A\cap D_\alpha = D_\alpha$. But $A \neq \emptyset$, so $A\cap D_\beta \neq \emptyset$ for some $\beta \Rightarrow A\cap D_\beta = D_\beta \Rightarrow x_0 \in A \Rightarrow A\cap D_\alpha \neq \emptyset \ \forall \alpha$
$\Rightarrow A \cap D_\alpha = D_\alpha \ \forall\alpha \Rightarrow A = \bigcup_{\alpha\in I} D_\alpha$. ◪

Pf of Thm 1. Follows easily from Lemma 1, which immediately tells us that distinct components are disjoint. To show that every point is in some component, consider the family of connected subsets that contain the point and take their union. ◪

Pf of Thm 2. We first show that components of G are open. Let $D \subseteq G$ be a component, $z_0 \in D$. Since G is open $\exists\, \varepsilon > 0$ s.t. $B(z_0, \varepsilon) \subseteq G$. But if $B(z_0,\varepsilon) \nsubseteq D$, then $B(z_0,\varepsilon)\cup D$, which is connected by Lemma 1, would properly contain D, contradicting that D is a component. Thus, D is open.

To see that the collection of components is countable, consider the Gaussian rationals $\{z_n = x_n + iy_n : x_n, y_n \in \mathbb{Q}\}$. ← rational #.

This is a countable, dense subset of $\mathbb{C}$. Since each component D_α is open, it must contain at least one (well, ∞ many) of the z_n's. Since each z_n can only belong to one component, we get a surjective map $\mathbb{N} \to \{D_\alpha\}$. Thus, the collection is countable

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